Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games

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1. Introduction

The concept of a perfect equilibrium point has been introduced in order to exclude the possibility that disequilibrium behavior is prescribed on unreached subgames [SELTEN 1965 and 1973]. Unfortunately this definition of perfectness does not remove all difficulties which may arise with respect to unreached parts of the game. It is necessary to reexamine the problem of defining a satisfactory non-cooperative equilibrium concept for games in extensive form. Therefore a new concept of a perfect equilibrium point will be introduced in this paper[2]).

In retrospect the earlier use of the word "perfect" was premature. Therefore a perfect equilibrium point in the old sense will be called "subgame perfect". The new definition of perfectness has the property that a perfect equilibrium point is always subgame perfect but a subgame perfect equilibrium point may not be perfect.

It will be shown that every finite extensive game with perfect recall has at least one perfect equilibrium point.

Since subgame perfectness cannot be detected in the normal form, it is clear that for the purpose of the investigation of the problem of perfectness, the normal form is an inadequate representation of the extensive form. It will be convenient to introduce an "agent normal form" as a more adequate representation of games with perfect recall.

2. Extensive Games with Perfect Recall

In this paper the words *extensive game* will always refer to a finite game in extensive form. A game of this kind can be described as a sextuple.

$$\Gamma = (K, P, U, C, p, h) \tag{1}$$

where the *constituents* K, P, U, A, p and h of Γ are as follows[3]):

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[2]) The idea to base the definition of a perfect equilibrium point on a model of slight mistakes as described in section 7 is due to JOHN C. HARSANYI. The author's earlier unpublished attempts at a formalization of this concept were less satisfactory. I am very grateful to JOHN C. HARSANYI who strongly influenced the content of this paper.

[3]) The notation is different from that used by KUHN [1953].

The game tree:

The game tree K is a finite tree with a distinguished vertex o, the *origin* of K. The sequence of vertices and edges which connects o with a vertex x is called the *path* to x. We say that x comes before y or that y comes after x if x is different from y and the path to y contains the path to x. An *endpoint* is a vertex z with the property that no vertex comes after z. The set of all endpoints is denoted by Z. A path to an endpoint is called a play. The edges are also called *alternatives*. An alternative at x is an edge which connects x with a vertex after x. The set of all vertices of K which are not endpoints, is denoted by X.

The player partition:

The player partition $P = (P_0, \ldots, P_n)$ partitions X into *player sets*. P_i is called player i's player set (Player 0 is the "random" player who represents the random mechanisms responsible for the random decisions in the game.) A player set may be empty. The player sets P_i with $i = 1, \ldots, n$ are called *personal player* sets.

The information partition:

For $i = 1, \ldots, n$ a subset u of P_i is called *eligible* (as an information set) if n is not empty, if every play intersects u at most once and if the number of alternatives at x is the same for every $x \in u$. A subset $u \in P_0$ is called *eligible* if it contains exactly one vertex. The *information partition* U is a refinement of the player partition P into eligible subsets u of the player sets. These sets u are called *information sets*. The information sets u with $u \subseteq P_i$ are called information sets of player i. The set of all information sets of player i is denoted by U_i. The information sets of player $1, \ldots, n$ are called *personal* information sets.

The choice partition:

For $u \in U$ let A_u be the set of all alternatives at vertices $x \in u$. We say that a subset c of A_u is *eligible* (as a choice) if it contains exactly one alternative at x for every vertex $x \in u$. The *choice partition* C partitions the set of all edges of K into eligible subsets c of the A_u with $u \in U$. These sets c are called *choices*. The choices c which are subsets of A_u are called choices *at* u. The set of all choices at u is denoted by C_u. A choice at a personal information set is called a personal choice. A choice which is not personal is a *random choice*. We say that the vertex x comes after the choice c if one of the edges in c is on the path to x. In this case we also say that c is *on* the path to x.

The probability assignment:

A probability distribution p_u over C_u is called *completely mixed* if it assigns a positive probability $p_u(c)$ to every $c \in C_u$. The *probability assignment* p is a function which assigns a completely mixed probability distribution p_u over C_u to every $u \in U_0$. (p specifies the probabilities of the random choices.)

The payoff function:

The *payoff function* h assigns a vector $h(z) = (h_1(z),...,h_n(z))$ with real numbers as components to every endpoint z of K. The vector $h(z)$ is called the *payoff vector* at z. The component $h_i(z)$ is player i's *payoff* at z.

Perfect recall:

An extensive game $\Gamma = (K,P,U,C,p,h)$ is called an extensive game with *perfect recall* if the following condition is satisfied for every player $i = 1,...,n$ and any two information sets u and v of the same player i: if one vertex $y \in v$ comes after a choice c at u then every vertex $x \in v$ comes after this choice c [4]).

Interpretation:

In a game with perfect recall a player i who has to make a decision at one of his information sets v knows which of his other information sets have been reached by the previous course of the play and which choices have been taken there. Obviously a player always must have this knowledge if he is a person with the ability to remember what he did in the past. Since game theory is concerned with the behavior of absolutely rational decision makers whose capabilities of reasoning and memorizing are unlimited, a game, where the players are individuals rather than teams, must have perfect recall.

Is there any need to consider games where the players are teams rather than individuals? In the following we shall try to argue that at least as far as strictly non-cooperative game theory is concerned the answer to this question is no. In principle it is always possible to model any given interpersonal conflict situation in such a way that every person involved is a single player. Several persons who form a team in the sense that all of them pursue the same goals can be regarded as separate players with identical payoff functions. Against this view one might object that a team may be united by more than accidentally identical payoffs. The team may be a preestablished coalition with special cooperative possibilities not open to an arbitrary collection of persons involved in the situation. This is not a valid objection. Games with preestablished coalitions of this kind are outside the framework of strictly non-cooperative game theory. In a strictly non-cooperative game the players do not have any means of cooperation or coordination which are not explicitly modelled as parts of the extensive form. If there is something like a preestablished coalition, then the members must appear as separate players and the special possibilities of the team must be a part of the structure of the extensive game.

In view of what has been said no room is left for strictly non-cooperative extensive games without perfect recall. In the framework of strictly non-cooperative game theory such games can be rejected as misspecified models of interpersonal conflict situations.

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[4]) The concept of perfect recall has been introduced by H. W. KUHN [1953].

3. Strategies, Expected Payoff and Normal Form

In this section several definitions are introduced which refer to an extensive game $\Gamma = (K,P,U,A,p,h)$.

Local strategies:

A local strategy b_{iu} at the information set $u \in U_i$ is a probability distribution over the set C_u of the choices at u; a probability $b_{iu}(c)$ is assigned to every choice c at u. A local strategy b_{iu} is called *pure* if it assigns 1 to one choice c at u and 0 to the other choices. Wherever this can be done without danger of confusion no distinction will be made between the choice c and the pure local strategy which assigns the probability 1 to c.

Behavior strategies:

A behavior strategy b_i of a personal player i is a function which assigns a local strategy b_{iu} to every $u \in U_i$. The set of all behavior strategies of player i is denoted by B_i.

Pure strategies:

A *pure strategy* π_i of player i is a function which assigns a choice c at u (a pure local strategy) to every $u \in U_i$. Obviously a pure strategy is a special behavior strategy. The set of all pure strategies of player i is denoted by Π_i.

Mixed strategies:

A mixed strategy q_i of player i is a probability distribution over Π_i; a probability $q_i(\pi_i)$ is assigned to every $\pi_i \in \Pi_i$. The set of all mixed strategies q_i of player i is denoted by Q_i. Wherever this can be done without danger of confusion no distinction will be made between the pure strategy π_i and the mixed strategy q_i which assigns 1 to π_i. Pure strategies are regarded as special cases of mixed strategies.

Behavior strategy mixtures:

A behavior strategy mixture s_i for player i is a probability distribution over B_i which assigns positive probabilities $s_i(b_i)$ to a finite number of elements of B_i and zero probabilities to the other elements of B_i. No distinction will be made between the behavior strategy b_i and the behavior strategy mixture which assigns 1 to b_i. The set of all behavior strategy mixtures of player i is denoted by S_i. Obviously pure strategies, mixed strategies and behavior strategies can all be regarded as special behavior strategy mixtures.

Combinations:

A combination $s = (s_1,\ldots,s_n)$ *of behavior strategy mixtures* is an n-tuple of behavior strategy mixtures $s_i \in S_i$, one for each personal player. *Pure strategy combinations* $\pi = (\pi_1,\ldots,\pi_n)$, *mixed strategy combinations* and *behavior strategy combinations* are defined analogously.

Realization probabilities:

A player i who plays a behavior strategy mixture s_i behaves as follows: He first employs a random mechanism which selects one of the behavior strategies b_i with the probabilities $s_i(b_i)$. He then in the course of the play at every $u \in U_i$ which is reached by the play selects one of the choices c at u with the probabilities $b_{iu}(c)$. Let $s = (s_1, \ldots, s_n)$ be a combination of behavior strategy mixtures. On the assumption that the s_i are played by the players we can compute a *realization probability* $\rho(x,s)$ of x under s for every vertex $x \in K$. This probability $\rho(x,s)$ is the probability that x is reached by the play, if s is played. Since these remarks make it sufficiently clear, how $\rho(x,s)$ is defined, a more precise definition of $\rho(x,s)$ will not be given here.

Expected payoffs:

With the help of the realization probabilities an *expected payoff vector* $H(s) = (H_1(s), \ldots, H_n(s))$ can be computed as follows:

$$H(s) = \sum_{z \in Z} \rho(z,s) h(z) . \tag{2}$$

Since pure strategies, mixed strategies and behavior strategies are special cases of behavior strategy mixtures, the expected payoff definition (2) is applicable here, too.

Normal form:

A *normal form* $G = (\Pi_1, \ldots, \Pi_n; H)$ consists of n finite non-empty and pairwise non-intersecting *pure strategy sets* Π_i and an *expected payoff function* H defined on $\Pi = \Pi_1 x, \ldots, x \Pi_n$. The expected payoff function H assigns a payoff vector $H(\pi) = (H_1(\pi), \ldots, H_n(\pi))$ with real numbers as components to every $\pi \in \Pi$. For every extensive game Γ the pure strategy sets and the expected payoff function defined above generate the *normal form of* Γ.

In order to compute the expected payoff vector for a mixed strategy combination, it is sufficient to know the normal form of Γ. The same is not true for combinations of behavior strategies. As we shall see, in the transition from the extensive form to the normal form some important information is lost.

4. KUHN's Theorem

H. W. KUHN [1953, p. 213] has proved an important theorem on games with perfect recall. In this section KUHN's theorem will be restated in a slightly changed form. For this purpose some further definitions must be introduced. As before, these definitions refer to an extensive game $\Gamma = (K, P, U, A, p, h)$.

Notational convention:

Let $s = (s_1, \ldots, s_n)$ be a combination of behavior strategy mixtures and let t_i be a behavior strategy mixture for player i.

The combination $(s_1, \ldots, s_{i-1}, t_i, s_{i+1}, \ldots, s_n)$ which results from s, if s_i is replaced by t_i and the other components of s remain unchanged, is denoted by s/t_i. The same notational convention is also applied to other types of strategy combinations.

Realization equivalence:

Let s_i' and s_i'' be two behavior strategy mixtures for player i. We say that s_i' and s_i'' are *realization equivalent* if for every combination s of behavior strategy mixtures we have:

$$\rho(x, s/s_i') = \rho(x, s/s_i'') \quad \text{for every} \quad x \in K. \tag{3}$$

Payoff equivalence:

Let s_i' and s_i'' be two behavior strategy mixtures for player i. We say that s_i' and s_i'' are *payoff equivalent* if for every combination s of behavior strategy mixtures we have

$$H(s/s_i') = H(s/s_i''). \tag{4}$$

Obviously s_i' and s_i'' are payoff equivalent if they are realization equivalent, since (3) holds for the endpoints z.

Theorem 1 (KUHN's theorem):

In every extensive game with perfect recall a realization equivalent behavior strategy b_i can be found for every behavior strategy mixture s_i of a personal player i.

In order to prove this theorem we introduce some further definitions.

Conditional choice probabilities:

Let $s = (s_1, \ldots, s_n)$ be a combination of behavior strategy mixtures and let x be a vertex in an information set u of a personal player i, such that $\rho(x, s) > 0$. For every choice c at u we define a *conditional choice probability* $\mu(c, x, s)$. The choice c contains an edge e at x; this edge e connects x with another vertex y. The probability $\mu(c, x, s)$ is computed as follows:

$$\mu(c, x, s) = \frac{\rho(y, s)}{\rho(x, s)}. \tag{5}$$

The probability $\mu(c, x, s)$ is the conditional probability that the choice c will be taken if s is played and x has been reached.

Lemma 1:

In every extensive game Γ (with or without perfect recall) on the region of those triples (c, x, s) where the conditional choice probability $\mu(c, x, s)$ is defined the conditional choice probabilities $\mu(c, x, s)$ with $x \in u \in U_i$ do not depend on the components s_j of s with $i \neq j$.

Proof:

Let $b_i^1, \ldots, b_i^k$ be the behavior strategies which are selected by s_i with positive probabilities $s_i(b_i^j)$. For $\rho(x,s) > 0$ an outside observer, who knows that c has been reached by the play but does not know which of the b_i^j has been selected before the beginning of the game, can use this knowledge in order to compute posterior probabilities $t_i(b_i^j)$ from the prior probabilities $s_i(b_i^j)$. The posterior probability $t_i(b_i^j)$ is proportional to $s_i(b_i^j)$ multiplied by the product of all probabilities assigned by b_i^j to choices of player i on the path to x. Obviously the $t_i(b_i^j)$ depend on s_i but not on the other components of s. The conditional choice probability $\mu(c,x,s)$ can be written as follows:

$$\mu(c,x,s) = \sum_{j=1}^{k} t_i(b_i^j) b_{iu}^j(c). \tag{6}$$

This shows that $\mu(c,x,s)$ does not depend on the s_j with $i \neq j$.

Lemma 2:

In every extensive game Γ with perfect recall, on the region of those triples (c,x,s) where the conditional choice probability $\mu(c,x,s)$ is defined, we have

$$\mu(c,x,s) = \mu(c,y,s) \quad \text{for} \quad x \in u \quad \text{and} \quad y \in u. \tag{7}$$

Proof:

In a game with perfect recall for $x \in u$, $y \in u$ and $u \in U_i$ player i's choices on the path to x are the same choices as his choices on the path to y. (This is not true for games without perfect recall). Therefore at x and y the posterior probabilities for the behavior strategies b_i^j occurring in player i's behavior strategy mixture s_i are the same at both vertices. Consequently (7) follows from (6).

Proof of KUHN's *theorem:*

In view of lemma 1 and lemma 2 the conditional choice probabilities at the vertices x in the player set P_i of a personal player can be described by a function $\mu_i(c,u,s_i)$ which depends on his behavior strategy mixture s_i and the information set u with $x \in u$.

With the help of $\mu_i(c,u,s_i)$ we construct the behavior strategy b_i whose existence is asserted by the theorem. If for at least one $s = (s_1, \ldots, s_n)$ with s_i as component we have $\mu(x,s) > 0$ for some $x \in u$, we define

$$b_{iu}(c) = \mu_i(c,u,s_i). \tag{8}$$

The construction of b_i is completed by assigning arbitrary local strategies b_{iu} to those $u \in U_i$ where no such s can be found.

It is clear that this behavior strategy b_i and the behavior strategy mixture s_i are realization equivalent.

The significance of KUHN's *theorem:*

The theorem shows that in the context of extensive games with perfect recall one can restrict one's attention to behavior strategies. Whatever a player can achieve by a mixed strategy or a more general behavior strategy mixture can be achieved by the realization equivalent and therefore also payoff equivalent behavior strategy whose existence is secured by the theorem.

5. Subgame Perfect Equilibrium Points

In this section we shall introduce some further definitions which refer to an extensive game $\Gamma = (K, P, U, A, p, h)$ with perfect recall. In view of KUHN's theorem only behavior strategies are important for such games. Therefore the concepts of a best reply and an equilibrium point are formally introduced for behavior strategies only.

Best reply:

Let $b = (b_1, ..., b_n)$ be a combination of behavior strategies for Γ. A behavior strategy $\tilde{b}_i$ of player i as a *best reply* to b if we have

$$H_i(b/\tilde{b}_i) = \max_{b_i' \in B_i} H_i(b/b_i') . \tag{9}$$

A combination of behavior strategies $\tilde{b} = (\tilde{b}_1, ..., \tilde{b}_n)$ is called a *best reply* to b if for $i = 1, ..., n$ the behavior strategy $\tilde{b}_i$ is a best reply to b.

Equilibrium point:

A behavior strategy combination $b^* = (b_1^*, ..., b_n^*)$ is called an *equilibrium point* if b^* is a best reply to itself.

Remark:

The concepts of a best reply and an equilibrium point can be defined analogously for behavior strategy mixtures. In view of KUHN's theorem it is clear that for games with perfect recall an equilibrium point in behavior strategies is a special case of an equilibrium point in behavior strategy mixtures. The existence of an equilibrium point in behavior strategies for every extensive game with perfect recall is an immediate consequence of KUHN's theorem together with NASH's well known theorem on the existence of an equilibrium point in mixed strategies for every finite game [NASH, 1951].

Subgame:

Let $\Gamma = (K, P, U, A, p, h)$ be an extensive game with or without perfect recall. A subtree K' of K consists of a vertex x of K together with all vertices after x and all edges of K connecting vertices of K'. A subtree K' is called *regular* in Γ, if every information set in Γ, which contains at least one vertex of K', does not contain any vertices outside of K'. For every regular subtree K' a *subgame* $\Gamma' =$

(K', P', U', A', p', h') is defined as follows: P', U', A', p' and h' are the restrictions of the partitions P, U, A and the functions p and h to K'.

Induced strategies:

Let Γ' be a subgame of Γ and let $b = (b_1, \ldots, b_n)$ be a behavior strategy combination for Γ. The restriction of b_i to the information sets of player i in Γ' is a strategy b_i' of player i for Γ'. This strategy b_i' is called *induced* by b_i on Γ' and the behavior strategy combination $b' = (b_1', \ldots, b_n')$ defined in this way is called *induced* by b on Γ'.

Subgame perfectness:

A *subgame perfect* equilibrium point $b^* = (b_i^*, \ldots, b_n^*)$ of an extensive game Γ is an equilibrium point (in behavior strategies) which induces an equilibrium point on every subgame of Γ.

6. A Numerical Example

The definition of a subgame perfect equilibrium point excludes some cases of intuitively unreasonable equilibrium points for extensive games. In this section we shall present a numerical example which shows that not every intuitively unreasonable equilibrium point is excluded by this definition. The discussion of the example will exhibit the nature of the difficulty.

The numerical example is the game of figure 1. Obviously this game has no subgames. Every player has exactly one information set. The game is a game with perfect recall.
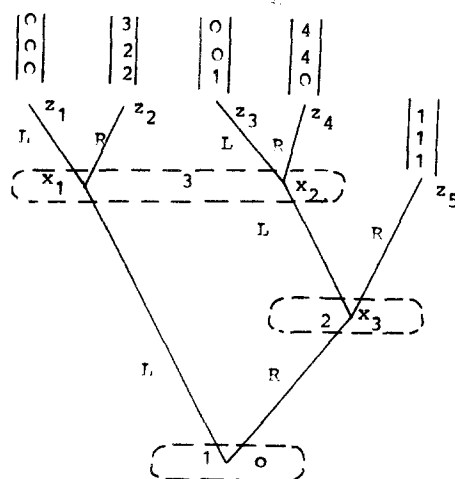

Fig. 1. A numerical example. Information sets are represented by dashed lines. Choices are indicated by the letters L and R (standing for "left" and "right"). Payoff vectors are indicated by column vectors above the corresponding endpoints

Since every player has two choices, L and R, a behavior strategy of player i can be characterized by the probability with which he selects R. The symbol p_i will be used for this probability. A combination of behavior strategies is represented by a triple (p_1, p_2, p_3).

As the reader can verify for himself without much difficulty the game of figure 1 has the following two types of equilibrium points:

$$\text{Type 1: } p_1 = 1, \ p_2 = 1, \ 0 \le p_3 \le \tfrac{1}{4}$$
$$\text{Type 2: } p_1 = 0, \ \tfrac{1}{3} \le p_2 \le 1, \ p_3 = 1 \, .$$

Consider the equilibrium points of type 2. Player 2's information set is not reached, if an equilibrium point of this kind is played. Therefore his expected payoff does not depend on his strategy. This is the reason why his equilibrium strategy is best reply to the equilibrium strategies of the other players.

Now suppose that the players believe that a specific type 2 equilibrium point, say $(0, 1, 1)$ is the rational way to play the game. Is it really reasonable to believe that player 2 will choose R if he is reached? If he believes that player 3 will choose R as prescribed by the equilibrium point, then it is better for him to select L where he will get 4 instead of R where he will get 1. The same reasoning applies to the other type 2 equilibrium points, too.

Clearly, the type 2 equilibrium points cannot be regarded as reasonable. Player 2's choices should not be guided by his payoff expectations in the whole game but by his conditional payoff expectations at x_3. The payoff expectation in the whole game is computed on the assumption that player 1's choice is L. At x_3 this assumption has been shown to be wrong. Player 2 has to assume that player 1's choice was R.

For every strategy combination (p_1, p_2, p_3) it is possible to compute player 2's conditional payoff expectations for his choices L and R on the assumption that his information set has been reached. The same cannot be done for player 3. Player 3's information set can be reached in two ways. Consider an equilibrium point of type 1, e.g. the equilibrium point $(1, 1, 0)$. Suppose that $(1, 1, 0)$ is believed to be the rational way to play the game and assume that contrary to the expectations generated by this belief, player 3's information set is reached. In this case player 3 must conclude that either player 1 or player 2 must have deviated from the rational way of playing the game but he does not know which one. He has no obvious way of computing a conditional probability distribution over the vertices in his information set, which tells him, with which probabilities he is at x_1 and at x_2 if he has to make his choice.

In the next section a model will be introduced which is based on the idea that with some very small probability a player will make a mistake. These mistake probabilities do not directly generate a conditional probability distribution over the vertice of player 3's information set. As we shall see in section 8 the introduction

of slight mistakes may lead to a strategic situation where the rational strategies add some small voluntary deviations to the mistakes.

7. A Model of Slight Mistakes

There cannot be any mistakes if the players are absolutely rational. Nevertheless, a satisfactory interpretation of equilibrium points in extensive games seems to require that the possibility of mistakes is not completely excluded. This can be achieved by a point of view which looks at complete rationality as a limiting case of incomplete rationality.

Suppose that the personal players in an extensive game Γ with perfect recall are subject to a slight imperfection of rationality of the following kind. At every information set u there is a small positive probability ε_u for the breakdown of rationality. Whenever rationality breaks down, every choice c at u will be selected with some positive probability q_c which may be thought of as determined by some unspecified psychological mechanism. Each of the probabilities ε_u and q_c is assumed to be independent of all the other ones.

Suppose that the rational choice at u is a local strategy which selects c with probability p_c. Then the total probability of the choice c will be

$$\hat{p}_c = (1 - \varepsilon_u)p_c + \varepsilon_u q_c . \tag{4}$$

The introduction of the probabilities ε_u and q_c transforms the original game into a changed game $\hat{\Gamma}$ where the players do not completely control their choices. A game of this kind will be called a *perturbed game* of Γ.

Obviously, it is not important whether the p_c or the $\hat{p}_c$ are considered to be the strategic variables of the perturbed game $\hat{\Gamma}$. In the following we shall take the latter point of view. This means that in $\hat{\Gamma}$ every player i selects a behavior strategy which assigns probability distributions over the choices c at u to the information sets u of player i in such a way that the probability $\hat{p}_c$ assigned to a choice c at u always satisfies the following condition:

$$\hat{p}_c \geq \varepsilon_u q_c . \tag{10}$$

The probability $\hat{p}_c$ is also restricted by the upper bound $1 - \varepsilon_u(1 - q_c)$; it is not necessary to introduce this upper bound explicitly since it is implied by the lower bounds on the probabilities of the other choices at the same information set. With the help of the notation

$$\eta_c = \varepsilon_u q_c \tag{11}$$

condition (10) can be rewritten as follows:

$$\hat{p}_c \geq \eta_c \quad \text{for every personal choice } c . \tag{12}$$

Consider a system of positive constants η_c for the personal choices c in Γ such that

$$\sum_{c\,at\,u} \eta_c < 1 \,. \tag{13}$$

Obviously for every system of this kind we can determine positive probabilities ε_u and q_c which generate a perturbed game $\hat{\Gamma}$ whose conditions (10) coincide with the conditions (12). Therefore we may use the following definition of a perturbed game.

Definition:

A *perturbed game* $\hat{\Gamma}$ is a pair (Γ, η) where Γ is an extensive game with perfect recall and η is a function which assigns a positive probability η_c to every personal choice c in Γ such that (13) is satisfied.

The probabilities η_c are called minimum probabilities. For every choice c at a personal information set u define

$$\mu_c = 1 + \eta_c - \sum_{c'\,at\,u} \eta_{c'} \tag{14}$$

obviously μ_c is the upper bound of $\hat{p}_c$ implied by the conditions (7). This probability μ_c is called the *maximum probability* of c.

Strategies:

A *local strategy* for the perturbed game $\hat{\Gamma} = (\Gamma, \eta)$ is a local strategy for Γ which satisfies the conditions (12). A *behavior strategy* of player i in $\hat{\Gamma}$ is a behavior strategy of player i in Γ which assigns local strategies for $\hat{\Gamma}$ to the information sets of player i. The set of all behavior strategies of player i for $\hat{\Gamma}$ is denoted by $\hat{B}_i$. A *behavior strategy combination* for $\hat{\Gamma}$ is a behavior strategy combination $\hat{b} = (\hat{b}_1, ..., \hat{b}_n)$ for Γ whose components are behavior strategies for $\hat{\Gamma}$. The set of all behavior strategy combinations for $\hat{\Gamma}$ is denoted by $\hat{B}$.

Best replies:

Let $b = (b_1, ..., b_n)$ be a behavior strategy combination for $\hat{\Gamma}$. A behavior strategy $\tilde{b}_i$ of player i for $\hat{\Gamma}$ is called a *best reply to b in* $\hat{\Gamma}$ if we have

$$H_i(b/\tilde{b}_i) = \max_{b_i' \in \hat{B}_i} H_i(b/b_i') \,. \tag{15}$$

A behavior strategy combination $\tilde{b} = (\tilde{b}_1, ..., \tilde{b}_n)$ for $\hat{\Gamma}$ is called a *best reply to b in* $\hat{\Gamma}$ if every component $\tilde{b}_i$ of b_i is a best reply to b in $\hat{\Gamma}$.

Equilibrium point:

An equilibrium point of $\hat{\Gamma}$ is a behavior strategy combination for $\hat{\Gamma}$ which is a best reply to itself in $\hat{\Gamma}$.

Remark:

Note that there is a difference between a best reply in Γ and a best reply in $\hat{\Gamma}$. The strategy sets $\hat{B}_i$ are subsets of the strategy sets B_i. Pure strategies are not available in $\hat{\Gamma}$.

8. Perfect Equilibrium Points

The difficulties which should be avoided by a satisfactory definition of a perfect equilibrium point are connected to unreached information sets. There cannot by any unreached information sets in the perturbed game. If b is a behavior strategy combination for the perturbed game then the realization probability $\rho(x, b)$ is positive for every vertex x of K. This makes it advantageous to look at a game Γ as a limiting case of perturbed games $\hat{\Gamma} = (\Gamma, \eta)$. In the following a perfect equilibrium point will be defined as a limit of equilibrium points for perturbed games.

Sequences of perturbed games:

Let Γ be an extensive game with perfect recall. A sequence $\hat{\Gamma}^1, \hat{\Gamma}^2, \ldots$ where for $k = 1, 2, \ldots$ the game $\hat{\Gamma}^k = (\Gamma, \eta^k)$ is a perturbed game of Γ, is called a test sequence for Γ, if for every choice c of the personal players in Γ the sequence of the minimum probabilities η_c^k assigned to c by η^k converges to 0 for $k \to \infty$.

Let $\hat{\Gamma}^1, \hat{\Gamma}^2, \ldots$ be a test sequence for Γ. A behavior strategy combination b^* for Γ is called a *limit equilibrium point* of this test sequence if for $k = 1, 2, \ldots$ an equilibrium point $\hat{b}^k$ of $\hat{\Gamma}^k$ can be found such that for $k \to \infty$ the sequence of the $\hat{b}^k$ converges to b^*.

Lemma 3:

A limit equilibrium point b^* of a test sequence $\hat{\Gamma}^1, \hat{\Gamma}^2, \ldots$ for an extensive game Γ with perfect recall is an equilibrium point of Γ.

Proof:

The fact that the b^k are equilibrium points of the $\hat{\Gamma}^k$ can be expressed by the following inequalities

$$H_i(\hat{b}^k) \geq H_i(\hat{b}^k/b_i) \quad \text{for every} \quad b_i \in \hat{B}_i^k \quad \text{and for} \quad i = 1, \ldots, n. \tag{16}$$

Let B_i^m be the intersection of all $\hat{B}_i^k$ with $k \geq m$. For $k \geq m$ we have

$$H_i(\hat{b}^k) \geq H_i(\hat{b}^k/b_i) \quad \text{for every} \quad b_i \in B_i^m. \tag{17}$$

Since the expected payoff depends continuously on the behavior strategy combination this inequality remains valid if on both sides we take the limits for $k \to \infty$. This yields:

$$H_i(b^*) \geq H_i(b^*/b_i) \quad \text{for every} \quad b_i \in B_i^m. \tag{18}$$

Inequality (18) holds for every m. The closure of the union of all B_i^m is B_i. This together with the continuity of H_i yields:

$$H_i(b^*) \geq H_i(b^*/b_i) \quad \text{for every} \quad b_i \in B_i. \tag{19}$$

Inequality (19) shows that b is an equilibrium point of Γ.

Perfect equilibrium point:

Let Γ be an extensive game with perfect recall. A *perfect equilibrium point* of Γ is a behavior strategy combination $b^* = (b_1^*,...,b_n^*)$ for Γ with the property that for at least one test sequence $\hat{\Gamma}^1, \hat{\Gamma}^2,...$ the combination b^* is a limit equilibrium point of $\hat{\Gamma}^1, \hat{\Gamma}^2,...$.

Interpretation:

A limit equilibrium point b^* of a test sequence has the property that it is possible to find equilibrium points of perturbed games as close to b^* as desired. The definition of a perfect equilibrium point is a precise statement of the intuitive idea that a reasonable equilibrium point should have an interpretation in terms of arbitrarily small imperfections of rationality. A test sequence which has b^* as limit equilibrium point provides an interpretation of this kind. If b^* fails to be the limit equilibrium point of at least one test sequence b^* must be regarded as instable against very small deviations from perfect rationality.

Up to now it has not been shown that perfectness implies subgame perfectness. In order to do this we need a lemma on the subgame perfectness of equilibrium points for perturbed games.

Subgames of perturbed games:

Let $\hat{\Gamma} = (\Gamma, \eta)$ be a perturbed game of Γ. A *subgame* $\hat{\Gamma}' = (\Gamma', \eta')$ of $\hat{\Gamma}$ consists of a subgame Γ' of Γ and the restriction η' of η to the personal choices of Γ'. We say that $\hat{\Gamma}'$ is generated by Γ'. An equilibrium point $\hat{b}$ of $\hat{\Gamma}$ is called *subgame perfect* if an equilibrium point $\hat{b}'$ is induced on every subgame $\hat{\Gamma}'$ of $\hat{\Gamma}$.

Lemma 3:

Let Γ be an extensive game with perfect recall and let $\hat{\Gamma} = (\Gamma, \eta)$ be a perturbed game of Γ. Every equilibrium point of $\hat{\Gamma}$ (in behavior strategies) is subgame perfect.

Proof:

Let $\hat{b}'$ be the behavior strategy combination induced by an equilibrium point $\hat{b}$ of $\hat{\Gamma}$ on a subgame Γ' of Γ. Obviously $\hat{b}'$ is a behavior strategy combination for the subgame $\hat{\Gamma}' = (\Gamma', \eta')$ generated by Γ'. Suppose that $\hat{b}'$ fails to be an equilibrium point of $\hat{\Gamma}'$. It follows that for some personal player j a behavior strategy b_j' for $\hat{\Gamma}'$ exists, such that player j's expected payoff for $\hat{b}'/b_j'$ in $\hat{\Gamma}'$ is greater than his expected payoff for $\hat{b}'$ in $\hat{\Gamma}'$. Consider the behavior strategy b_j for $\hat{\Gamma}$ which agrees with b_j' on Γ' and with player j's strategy $\hat{b}_j$ in $\hat{b}$ everywhere else. Since the realization probabilities in $\hat{\Gamma}$ are always positive player j's expected payoff for $\hat{b}/b_j$ must be greater than his expected payoff for $\hat{b}$. Since a behavior strategy b_j with this property does not exist, $\hat{b}'$ is an equilibrium point of $\hat{\Gamma}'$.

Theorem 2:

Let Γ be an extensive game with perfect recall and let $\tilde{b}$ be a perfect equilibrium point of Γ. On every subgame Γ' of Γ a perfect equilibrium point $\tilde{b}'$ is induced by $\tilde{b}$ on Γ'.

Corollary:

Every perfect equilibrium point of an extensive game Γ with perfect recall is a subgame perfect equilibrium point of Γ.

Proof:

Let $\hat{\Gamma}^1, \hat{\Gamma}^2, \ldots$ be a test sequence for Γ which has $\hat{b}$ as limit equilibrium point. Let $\hat{b}^1, \hat{b}^2, \ldots$ be a sequence of equilibrium points $\hat{b}^k$ of $\hat{\Gamma}^k$. It follows from the subgame perfectness of the $\hat{b}^k$ that the subgames of $\hat{\Gamma}^k$ generated by Γ' form a test sequence for Γ' with $\tilde{b}'$ as a limit equilibrium point. Therefore $\tilde{b}'$ is a perfect equilibrium point of Γ'.

The corollary is an immediate consequence of the fact that a perfect equilibrium point is an equilibrium point. (See lemma 3.)

9. A Second Look at the Numerical Example

In this section we shall first look at a special test sequence of the numerical example of figure 1 in order to compute its limit equilibrium point. The way in which this limit equilibrium point is approached exhibits an interesting phenomenon which is important for the interpretation of perfect equilibrium points. Later we shall show that every equilibrium point of type 1 is perfect.

Let $\varepsilon_1, \varepsilon_2, \ldots$ be a monotonically decreasing sequence of positive probabilities with $\varepsilon_1 < \frac{1}{4}$ and $\varepsilon_k \to 0$ for $k \to \infty$. Let Γ be the game of figure 1. Consider the following test sequence $\hat{\Gamma}^1, \hat{\Gamma}^2, \ldots$ for Γ. For $k = 1, 2, \ldots$ the perturbed game $\hat{\Gamma}^k = (\Gamma, \eta^k)$ is defined by $\eta_c^k = \varepsilon_k$ for every choice c of Γ.

As in section 7 let p_i be the probability of player i's choice R. A behavior strategy combination can be represented by a triple $p = (p_1, p_2, p_3)$. The behavior strategy combinations for $\hat{\Gamma}^k$ are restricted by the condition

$$1 - \varepsilon_k \geq p_i \geq \varepsilon_k \quad \text{for} \quad i = 1, 2, 3. \tag{20}$$

As we shall see, the perturbed game $\hat{\Gamma}^k$ has only one equilibrium point $p^k = (p_1^k, p_2^k, p_3^k)$ whose components p_i^k are as follows:

$$p_1^k = 1 - \varepsilon_k \tag{21}$$

$$p_2^k = 1 - \frac{2\varepsilon_k}{1 - \varepsilon_k} \tag{22}$$

$$p_3^k = \tfrac{1}{4}. \tag{23}$$

Equilibrium property of p^k:

In the following it will be shown that p^k is an equilibrium point of $\hat{\Gamma}^k$. Let us first look at the situation of player 3. For any $p = (p_1, p_2, p_3)$ the realization probabilities $\rho(x_1, p)$ and $\rho(x_2, p)$ of the vertices x_1 and x_2 in the information set of player 3 are given by (24) and (25).

$$\rho(x_1, p) = 1 - p_1 \tag{24}$$

$$\rho(x_2, p) = p_1(1 - p_2). \tag{25}$$

Player 3's expected payoff under the condition that his information set is reached is $2\rho(x, p)$ if he takes his choice R and $\rho(x_2, p)$ if he takes his choice L. Therefore p_3 is a best reply to p in $\hat{\Gamma}^k$ if and only if the following is true:

$$p_3 = \varepsilon_k \qquad \text{for} \quad 2(1 - p_1) < p_1(1 - p_2) \tag{26}$$

$$\varepsilon_k \leq p_3 \leq 1 - \varepsilon_k \quad \text{for} \quad 2(1 - p_1) = p_1(1 - p_2) \tag{27}$$

$$p_3 = 1 - \varepsilon_k \qquad \text{for} \quad 2(1 - p_1) > p_1(1 - p_2). \tag{28}$$

In the case of p^k we have

$$\rho(x_1, p^k) = \varepsilon_k \tag{29}$$

$$\rho(x_2, p^k) = 2\varepsilon_k. \tag{30}$$

Therefore it follows by (27) that p_3^k is a best reply to p^k. Let us now look at the situation of player 2. Here we can see that p_2 is a best reply to p in $\hat{\Gamma}^k$ if and only if the following is true:

$$p_2 = \varepsilon_k \qquad \text{for} \quad p_3 > \tfrac{1}{4} \tag{31}$$

$$\varepsilon_k \leq p_2 \leq 1 - \varepsilon_k \quad \text{for} \quad p_3 = \tfrac{1}{4} \tag{32}$$

$$p_2 = 1 - \varepsilon_k \qquad \text{for} \quad p_3 < \tfrac{1}{4}. \tag{33}$$

p_2^k is best reply to p^k in view of (32).

p_1 is a best reply to p in $\hat{\Gamma}^k$ if and only if the following is true:

$$p_1 = \varepsilon_k \qquad \text{for} \quad 3p_3 > 4(1 - p_2)p_3 + p_2 \tag{34}$$

$$\varepsilon_k \leq p_1 \leq 1 - \varepsilon_k \quad \text{for} \quad 3p_3 = 4(1 - p_2)p_3 + p_2 \tag{35}$$

$$p_1 = 1 - \varepsilon_k \qquad \text{for} \quad 3p_3 < 4(1 - p_2)p_3 + p_2. \tag{36}$$

p_1^k is best reply to p^k in view of (36).

Uniqueness of the equilibrium point:

In the following it will be shown that p^k is the only equilibrium point of $\hat{\Gamma}^k$. We first exclude the possibility $p_3 \neq 1/4$. Suppose that p is an equilibrium point with $p_3 < 1/4$. It follows by (33) that we have $p_2 = 1 - \varepsilon_k$. Consequently $3p_3$

is smaller than p_2 and (36) yields $p_1 = 1 - \varepsilon_k$. Therefore (28) applies to p_3. We have $p_3 = 1 - \varepsilon_k$ contrary to the assumption $p_3 < 1/4$.

Now we suppose that p is an equilibrium point with $p_3 > 1/4$. Condition (31) yields $p_2 = \varepsilon_k$. In view of $1 - p_2 > 3/4$ condition (36) applies to p_1. It follows that (26) applies to p_3 contrary to the assumption $p_3 > 1/4$.

We know now that an equilibrium point p of $\hat{\Gamma}^k$ must have the property $p_3 = \frac{1}{4}$. Obviously (36) applies to an equilibrium point p. We must have $p_1 = 1 - \varepsilon_k$. Moreover neither (26) nor (28) are satisfied by p_3. Therefore in view of (27) an equilibrium point p has the following property:

$$2(1 - p_1) = p_1(1 - p_2). \tag{37}$$

This together with $p_1 = 1 - \varepsilon_k$ yields

$$p_2 = \frac{2\varepsilon_k}{1 - \varepsilon_k}. \tag{38}$$

Voluntary deviations from the limit equilibrium point:

For $k \to \infty$ the sequence p^k converges to $p^* = (1, 1, 1/4)$. This is the only limit equilibrium point of the test sequence $\hat{\Gamma}^1, \hat{\Gamma}^2, \dots$.

Note that p_1^k is as near as possible to $p_1^* = 1$ since p_1^k is the maximum probability $1 - \varepsilon_k$. Contrary to this p_2^k is not as near as possible to p_2^*. The probability p_2^* is smaller than $1 - \varepsilon_k$ by $\varepsilon_k(1 + \varepsilon_k)/(1 - \varepsilon_k)$. The rules of the perturbed game force player 2 to take his choice L with a probability of at least ε_k but to this minimum probability he adds the "voluntary" probability $\varepsilon_k(1 + \varepsilon_k)/(1 - \varepsilon_k)$. In this sense we can speak of a voluntary deviation from the limit equilibrium point.

The voluntary deviation influences the realization probabilities $\rho(x_1, p^k)$ and $\rho(x_2, p^k)$. The conditional probabilities for x_1 and x_2, if the information set of player 3 is reached by p^k, are $1/3$ and $2/3$ for every k. It is natural to think of these conditional probabilities as conditional probabilities for the limit equilibrium point p^*, too. The assumptions on the probabilities of slight mistakes which are embodied in the test sequence $\hat{\Gamma}^1, \hat{\Gamma}^2, \dots$ do not directly determine these conditional probabilities but indirectly via the equilibrium points p^k.

Perfectness of the equilibrium points of type 1:

In the following it will be shown that every equilibrium point of type 1 is perfect. Let $p^* = (1, 1, p_3^*)$ be one of these equilibrium points. We construct a test sequence $\hat{\Gamma}^1, \hat{\Gamma}^2, \dots$ with the property that p^* is a limit equilibrium point of $\hat{\Gamma}^1, \hat{\Gamma}^2, \dots$. Let $\varepsilon_1, \varepsilon_2, \dots$ be a decreasing sequence of positive numbers with $\varepsilon_1 < p_3^*/2$ and $\varepsilon_k \to 0$ for $k \to \infty$. The minimum probabilities η_c^k for the perturbed game $\hat{\Gamma}^k = (\Gamma, \eta^k)$ are defined as follows:

$$\eta_c^k = \begin{cases} \varepsilon_k & \text{if } c \text{ is a choice of player 1 or player 3} \\ \dfrac{2\varepsilon_k}{1 - \varepsilon_k} & \text{if } c \text{ is a choice of player 2.} \end{cases} \tag{39}$$

With the help of arguments similar to those which have been used in the sub-section "equilibrium property of p^{k}", it can be shown that for $k = 1,2,\ldots$ the following behavior strategy combination $\hat{p}^{k} = (\hat{p}_1^k, \hat{p}_2^k, \hat{p}_3^k)$ is an equilibrium point of $\hat{\Gamma}^k$

$$\hat{p}_1 = 1 - \varepsilon_k \tag{40}$$

$$\hat{p}_2 = 1 - \frac{2\varepsilon_k}{1 - \varepsilon_k} \tag{41}$$

$$\hat{p}_3 = p_3^\star. \tag{42}$$

The sequence $\hat{p}^1, \hat{p}^2, \ldots$ converges to $p^\star$. Therefore $p^\star$ is a perfect equilibrium point.

Imperfectness of the equilibrium points of type 2:

In the following it will be shown that the equilibrium points of type 2 fail to be perfect. Let $p^\star = (0, p_2^\star, 1)$ be an equilibrium point of type 2 and let $\hat{\Gamma}^1, \hat{\Gamma}^2, \ldots$ be a test sequence which has $p^\star$ as limit equilibrium point. Let $p^1, p^2, \ldots$ be a sequence of equilibrium points $p^\star$ of $\hat{\Gamma}^k$ which for $k \to \infty$ converges to $p^\star$. For every $\varepsilon > 0$ we can find a number $m(\varepsilon)$ such that for $k > m(\varepsilon)$ the following two conditions (a) and (b) are satisfied. (a) Every minimum probability η_c^k in $\hat{\Gamma}^k = (\Gamma, \eta^k)$ is smaller than ε. (b) For $i = 1,2,3$ we have $|p_i^\star - p_i^k| < \varepsilon$. For sufficiently small ε it follows from (a) and (b) that p_2^k is not a best reply to p^k; we must have $p_2 < \varepsilon$ for player 2's best reply to p^k and p_2^k cannot be below $1/3$ by more than ε. This shows that $p^\star$ cannot be the limit equilibrium point of a test sequence.

10. A Decentralization Property of Perfect Equilibrium Points

In this section it will be shown that the question whether a given behavior strategy combination is a perfect equilibrium point or not, can be decided locally at the information sets of the game. The concept of a local equilibrium point will be introduced which is defined by conditions on the local strategies. As we shall see, in perturbed games these local conditions are equivalent to the usual global equilibrium conditions. On the basis of this result a decentralized description of a perfect equilibrium point will be developed.

Notational convention:

Let Γ be an extensive game and let b_i be a behavior strategy of a personal player i in Γ. Let b'_{iu} be a local strategy at an information set u of player i. The notation b_i/b'_{iu} is used for that behavior strategy which results from b_i if the local strategy assigned by b_i to u is changed to b'_{iu} whereas the local strategies assigned by b_i to other information sets remain unchanged. Let $b = (b_1, \ldots, b_n)$ be a behavior strategy combination. The notation b/b'_{iu} is used for the behavior strategy combination b/b'_i with $b'_i = b_i/b'_{iu}$. The set of all local strategies at u is denoted by B_{iu}.

Local best replies:

Let $b = (b_1, \ldots, b_n)$ be a behavior strategy combination for an extensive game Γ and let $\tilde{b}_{iu}$ be a local strategy at an information set u of a personal player i. The local strategy $\tilde{b}_{iu}$ is called a *local best reply to b in Γ* if we have

$$H_i(b/\tilde{b}_{iu}) = \max_{b'_{iu} \in B_{iu}} H_i(b/b'_{iu}) . \tag{43}$$

Local best replies in a perturbed game $\hat{\Gamma} = (\Gamma, \eta)$ are defined analogously: $\tilde{b}_{iu}$ is a *local best reply to b in $\hat{\Gamma}$* if we have

$$H_i(b/\tilde{b}_{iu}) = \max_{b'_{iu} \in \hat{B}_{iu}} H_i(b/b'_{iu}) \tag{44}$$

where $\hat{B}_{iu}$ is the set of all local strategies at u for $\hat{\Gamma}$.

Conditional realization probabilities:

Let $\hat{\Gamma} = (\Gamma, \eta)$ be a perturbed game of an extensive game Γ with perfect recall. For every information set u of a personal player i and every behavior strategy combination $b = (b_1, \ldots, b_n)$ for $\hat{\Gamma}$ we define a *conditional realization probability* $\mu(x,b)$:

$$\mu(x,b) = \frac{\rho(x,b)}{\sum\limits_{y \in u} \rho(y,b)} . \tag{45}$$

Obviously $\rho(x,b)$ is the conditional probability that x is reached by the play if b is played and u is reached. Since $\rho(x,b)$ is positive for every vertex x, the conditional realization probability $\mu(x,b)$ is defined for every vertex x. Let x be a vertex and let z be an endpoint after x. We define a second type of *conditional realization probability* $\mu(x,z,b)$ which is the probability that z will be reached if b is played and x has been reached. Obviously we have

$$\mu(x,z,b) = \frac{\rho(z,b)}{\rho(x,b)} . \tag{46}$$

Conditional expected payoff:

For every information set u of a personal player i in a perturbed game $\Gamma = (\hat{\Gamma}, \eta)$ of an extensive game Γ with perfect recall we define a *conditional expected payoff* function H_{iu} for player i at u:

$$H_{iu}(b) = \sum_{x \in u} \mu(x,b) \sum_{z \text{ after } x} \mu(x,z,b) h(z) . \tag{47}$$

$H_{iu}(b)$ is the conditional expectation of player i's payoff under the condition that b is played and u is reached by the play.

Lemma 4:

Let $b = (b_1, \ldots, b_n)$ be a behavior strategy combination for a perturbed game $\hat{\Gamma} = (\Gamma, \eta)$ of an extensive game Γ with perfect recall. The conditional realization probabilities $\mu(x,b)$ do not depend on b_i.

Proof:

In a game with perfect recall the information sets u of a personal player i have the property that the same choices of player i are on every path to a vertex $x \in u$. Therefore $\mu(x,b)$ does not depend on b_i.

Lemma 5:

Let $b = (b_1, \ldots, b_n)$ be a behavior strategy combination for a perturbed game $\hat{\Gamma} = (\Gamma, \eta)$ of an extensive game Γ with perfect recall and let $\tilde{b}_{iu}$ be a local strategy for $\hat{\Gamma}$ at an information set u of a personal player i. The local strategy $\tilde{b}_{iu}$ is a local best reply to b in $\hat{\Gamma}$ if and only if the following is true:

$$H_{iu}(b/\tilde{b}_{iu}) = \max_{b'_{iu} \in \hat{B}_{iu}} H_{iu}(b/b'_{iu}). \tag{48}$$

Proof:

The assertion of the lemma follows from the fact that the local strategy at u does not influence the realization probabilities of endpoints which do not come after vertices of u.

Lemma 6:

Let $b = (b_1, \ldots, b_n)$ be a behavior strategy combination for a perturbed game $\Gamma = (\Gamma, \eta)$ of an extensive game Γ with perfect recall and let $\tilde{b}_i$ be a behavior strategy for a personal player i in $\hat{\Gamma}$. The behavior strategy $\tilde{b}_i$ is a best reply to b in $\hat{\Gamma}$ if and only if for every local strategy $\tilde{b}_{iu}$ assigned to an information set $u \in U_i$ by $\tilde{b}_i$ the local strategy $\tilde{b}_{iu}$ is a local best reply to $b/\tilde{b}_i$ in $\hat{\Gamma}$.

Proof:

Suppose that for some $u \in U_i$, the local strategy $\tilde{b}_{iu}$ is not a local best reply to $b/\tilde{b}_i$ in $\hat{\Gamma}$. Let b'_{iu} be a local best reply to $b/\tilde{b}_i$ at u in $\hat{\Gamma}$. According to the definition of a local best reply $b'_i = \tilde{b}_i/b'_{iu}$ yields a higher payoff for player i than $\tilde{b}$, if the other players use their strategies in b. Therefore $\tilde{b}_i$ cannot be a best reply to b in $\hat{\Gamma}$. It follows that $\tilde{b}_{iu}$ is a local best reply to $b/\tilde{b}_i$ in $\hat{\Gamma}$.

Assume that every $\tilde{b}_{iu}$ is a local best reply to $b/\tilde{b}_i$ in $\hat{\Gamma}$ and that $\tilde{b}_i$ is not a best reply to b in $\hat{\Gamma}$. The theorem is true if this assumption leads to a contradiction. Let b'_i be a best reply to b in $\hat{\Gamma}$ and let b'_{iu} be the local strategies assigned by b'_i to the information sets $u \in U_i$. Let V_i be the set of all information sets $u \in U_i$, where b_{iu} is different from b'_{iu}. Obviously V_i is not empty.

In a game with perfect recall an information set $u \in U_i$ either comes *after* another information set $v \in U_i$ in the sense that every vertex $x \in u$ comes after a vertex $y \in v$ or u contains no vertex x which comes after a vertex of v. Therefore V_i contains information sets v such that no information set $u \in V_i$ has vertices after vertices of v. Let v be an information set of this kind.

We can assume without loss of generality that $b''_i = b'_i/\tilde{b}_{iv}$ is not a best reply to b in $\hat{\Gamma}$. Should b''_i be a best reply to b in $\hat{\Gamma}$, then we can use b''_i instead of b'_i for the purpose of this proof. If the same problem arises again, we can **repeat**

the procedure if necessary several times until finally we find a best reply of player i in $\hat{\Gamma}$ which suits our purpose. Now we assume that b_i'' is not a best reply to b in $\hat{\Gamma}$. With the notation $b/b_i'/\tilde{b}_{iv}$ for b/b_i'' we can write

$$H_i(b/b_i'/\tilde{b}_{iv}) < H_i(b/b_i') . \tag{49}$$

In the following we shall show that b_{iv} is a local best reply to b/b_i' in $\hat{\Gamma}$. This is a contradiction to (49).

It follows by lemma 4 that we have

$$\mu(x,b/b_i'/b_{iv}) = \mu(x,b/\tilde{b}_i/b_{iv}) \tag{50}$$

for every $x \in v$ and every local strategy b_{iv} of player i at v. Moreover the information set v has been selected in such a way that b_i' and $\tilde{b}_i$ assign the same probabilities to choices at information sets u after v. Therefore we have

$$\mu(x,z,b/b_i'/b_{iv}) = \mu(x,z,b/\tilde{b}_i/b_{iv}) \tag{51}$$

for every local strategy b_{iv} at v and for every $x \in v$. (47) together with (50) and (51) yields

$$H_{iv}(b/b_i'/b_{iv}) = (H_{iv}(b/\tilde{b}_i/b_{iv}) . \tag{52}$$

Since $\tilde{b}_{iv}$ is a local best reply to $b/\tilde{b}_i$ it is a consequence of lemma 5 and equation (52) that $\tilde{b}_{iv}$ is a local best reply to b/b_i'. This contradiction to (49) completes the proof of lemma 6.

Local equilibrium points:

A behavior strategy combination $b^* = (b_1^*,...,b_n^*)$ for an extensive game Γ is called a *local equilibrium point* for Γ or for a perturbed game $\hat{\Gamma}$ of Γ if every local strategy b_{iu}^* which is assigned to an information set u by one of the b_i^* is a local best reply to b in Γ or $\hat{\Gamma}$, resp.

Lemma 7:

A behavior strategy combination $b^* = (b_1^*,...,b_n^*)$ for a perturbed game $\hat{\Gamma} = (\Gamma,\eta)$ of an extensive game Γ with perfect recall is an equilibrium point for $\hat{\Gamma}$, if and only if b^* is a local equilibrium point for $\hat{\Gamma}$.

Proof:

The lemma is an immediate consequence of lemma 6.

Local limit equilibrium points:

Let $\hat{\Gamma}^1, \hat{\Gamma}^2,...$ be a test sequence for an extensive game Γ with perfect recall. A behavior strategy combination $b^* = (b_1^*,...,b_n^*)$ for Γ is called a *local limit equilibrium point* of the test sequence $\hat{\Gamma}^1, \hat{\Gamma}^2,...$ if every $\hat{\Gamma}^k$ has a local equilibrium point b^k such that for $k \to \infty$ the sequence of the b^k converges to b^*.

Theorem 3:

A behavior strategy combination $b^* = (b_1^*,...,b_n^*)$ for an extensive game Γ with perfect recall is a perfect equilibrium point of Γ, if and only if for at least

one test sequence $\hat{\Gamma}^1, \hat{\Gamma}^2,\ldots$ for Γ the behavior strategy combination b^* is a local limit equilibrium point of the test sequence $\hat{\Gamma}^1, \hat{\Gamma}^2,\ldots$.

Proof:

The theorem is an immediate consequence of lemma 7 and the definition of a perfect equilibrium point.

11. The Agent Normal Form and the Existence of a Perfect Equilibrium Point

In this section the concept of an agent normal form will be introduced. The players of the agent normal form are the agents of the information sets described by H. W. KUHN [1953] in his interpretation of the extensive form. An agent receives the expected payoff of the player to whom he belongs. The agent normal form contains all the information which is needed in order to compute the perfect equilibrium points of the extensive game. With the help of the agent normal form one can prove the existence of perfect equilibrium points for extensive games with perfect recall.

The agent normal form:

Let Γ be an extensive game and let $u_1,\ldots,u_N$ be the information sets of the personal players in Γ. For $i = 1,\ldots,N$ let ϕ_i be the set C_{u_i} of all choices at u_i. In the following we shall define a normal form $G = (\phi_1,\ldots,\phi_N,E)$ where the players $1,\ldots,N$ are thought of as agents associated with the information sets $u_1,\ldots,u_N$. This normal form is called the *agent normal form* of Γ.

Let ϕ be the set of all pure strategy combinations $\varphi = (\varphi_1,\ldots,\varphi_n)$ for G. For every $\varphi \in \phi$ the expected payoff vector $E(\varphi) = (E_1(\varphi),\ldots,E_n(\varphi))$ is defined as follows: Let $\pi = (\pi_1,\ldots,\pi_n)$ be the pure strategy combination for Γ whose components assign the choice $\varphi_j \in \phi_j$ to every information set u_j. For this π we have

$$E_i(\varphi) = H_j(\pi) \quad \text{for} \quad u_i \in U_j. \tag{53}$$

The expected payoff function E is extended to the mixed strategy combinations $q = (q_1,\ldots,q_N)$ of G in the usual way.

Induced strategy combinations:

Let $b = (b_1,\ldots,b_n)$ be a behavior strategy combination for Γ and let $q = (q_1,\ldots,q_N)$ be a mixed strategy combination for the agent normal form G of Γ. We say that q is *induced* by b on G and that b is *induced* on Γ by q if for $i = 1,\ldots,N$ the mixed strategy q_i is the same probability distribution over R_i as the local strategy assigned to u_i by the relevant component of b. Obviously this use of the word "induced" defines a one-to-one mapping between the behavior strategy combination b of Γ and the mixed strategy combinations q of G.

Perturbed agent normal forms:

Let G be a normal form $G = (\phi_1,\ldots,\phi_N,E)$ and let η be a function which assigns positive minimum probabilities η_c to every $c \in \phi_i$ with $i = 1,\ldots,N$, subject to the restriction

$$\sum_{c \in \phi_i} \eta_c < 1 \,. \tag{54}$$

The pair $\hat{G} = (G, \eta)$ is called a *perturbed normal form* of G. A mixed strategy q_i for G is a *mixed strategy for* $G = (G, \eta)$ if q_i satisfies the following condition:

$$q_i(c) \geq \eta_c \quad \text{for every} \quad c \in \phi_i \,. \tag{55}$$

A mixed strategy combination $q = (q_1, \ldots, q_N)$ is called a *mixed strategy combination for* $\hat{G} = (G, \eta)$ if for $i = 1, \ldots, N$ the mixed strategy q_i is a mixed strategy for $\hat{G}$. The set of all mixed strategies q_i of player i in $\hat{G}$ is denoted by $\hat{Q}_i$.

Let Γ be an extensive game and let G be the agent normal form of Γ. Obviously a behavior strategy combination for the perturbed game $\hat{\Gamma} = (\Gamma, \eta)$ is induced on Γ by every mixed strategy combination for the perturbed normal form $\hat{G} = (G, \eta)$ and vice versa. We call $\hat{G}$ the *perturbed agent normal form* of $\hat{\Gamma}$.

Equilibrium points:

A mixed strategy $\tilde{q}_i$ of a player i in a perturbed normal form $\hat{G} = (G, \eta)$ is called a *best reply to $\hat{G}$* to the mixed strategy combination $q = (q_1, \ldots, q_N)$ for $\hat{G}$ if we have

$$E_i(q/\tilde{q}_i) = \max_{q_i' \in \hat{Q}_i} E_i(q/q_i') \,. \tag{56}$$

A mixed strategy combination $\tilde{q} = (\tilde{q}_1, \ldots, \tilde{q}_N)$ is called a *best reply to q in $\hat{G}$*, if every $\tilde{q}_i$ in $\tilde{q}$ is a best reply to q in $\hat{G}$. A mixed strategy combination q^* for $\hat{G}$ is called an *equilibrium point of $\hat{G}$*, if q^* is a best reply to itself in $\hat{G}$.

Lemma 8:

Let $\hat{G} = (G, \eta)$ be the perturbed agent normal form of the perturbed game $\hat{\Gamma} = (\Gamma, \eta)$ of an extensive game Γ with perfect recall. An equilibrium point of $\hat{\Gamma}$ is induced on Γ by every equilibrium point of $\hat{G}$ and an equilibrium point of $\hat{G}$ is induced on G by every equilibrium point of $\hat{\Gamma}$.

Proof:

It is clear that a local best reply in $\hat{\Gamma}$ corresponds to a best reply in $\hat{G}$. Therefore the assertion follows by lemma 7.

Perfect equilibrium points:

A *test sequence* $\hat{G}^1, \hat{G}^2, \ldots$ for a normal form $G = (\phi_1, \ldots, \phi_N, E)$ is a sequence of perturbed normal forms $\hat{G}^k = (G, \eta^k)$ of G such that for $k \to \infty$ the sequence of the η_c^k converges to 0 for every c in the sets R_i. A *limit equilibrium point* q^* of a test sequence $\hat{G}^1, \hat{G}^2, \ldots$ is a mixed strategy combination for G, such that there is at least one sequence $q^1, q^2, \ldots$ of equilibrium points q^k for $\hat{G}^k$ which for $k \to \infty$ converges to q^*. A *perfect equilibrium point* of G is a mixed strategy combination q^* for G which is a limit equilibrium point of at least one test sequence $\hat{G}^1, \hat{G}^2, \ldots$ for G.

Lemma 9:

A limit equilibrium point q^* of a test sequence $\hat{G}^1, \hat{G}^2, \ldots$ for a normal form G is an equilibrium point of G.

Proof:

The proof is omitted here since it is completely analogous to the proof of lemma 3.

Theorem 4:

Let Γ be an extensive game with perfect recall and let G be the agent normal form of Γ. A perfect equilibrium point of Γ is induced on Γ by every perfect equilibrium point of G and a perfect equilibrium point of G is induced on G by every perfect equilibrium point of Γ.

Proof:

It follows by lemma 8 that a one-to-one relationship between the test sequences for Γ and for G can be established where a perturbed game of Γ corresponds to its perturbed agent normal form. Therefore a limit equilibrium point of one of both sequences induces a limit equilibrium point of the other one.

Existence of perfect equilibrium points:

In the following it will be shown that every extensive game Γ with perfect recall has at least one perfect equilibrium point. In order to prove this, we make use of theorem 4.

Theorem 5:

Every normal form G has at least one perfect equilibrium point.

Proof:

A perturbed normal form $\hat{G} = (G, \eta)$ satisfies well known sufficient conditions for the existence of an equilibrium point in mixed strategies [see e.g. BURGER, 1958, p. 35, Satz 2]. Therefore every perturbed normal form $\hat{G}^k$ in a test sequence $\hat{G}^1, \hat{G}^2, \ldots$ for G has an equilibrium point q^k. Since the set of all mixed strategy combinations is a closed and bounded subset of an eucledian space, the sequence $q^1, q^2, \ldots$ has an accumulation point q^*. The sequence $q^1, q^2, \ldots$ has a subsequence which converges to q^*. The corresponding subsequence of the test sequence $\hat{G}^1, \hat{G}^2, \ldots$ is a test sequence with the limit equilibrium point q^*. Therefore q^* is a perfect equilibrium point of G.

Theorem 6:

Every extensive game Γ with perfect recall has at least one perfect equilibrium point.

Proof:

In view of theorem 5 the agent normal form of Γ has a perfect equilibrium point. It follows by theorem 4 that Γ has a perfect equilibrium point.

12. Characterization of Perfect Equilibrium Points as Best Replies to Substitute Sequences

In this section it will be shown that the definition of a perfect equilibrium point as a limit equilibrium point of a test sequence is equivalent to another definition which is more advantageous from the point of view of mathematical simplicity. In view of theorem 4 we can restrict our attention to perfect equilibrium points for normal forms. It is sufficient to analyse the agent normal form if one wants to find the perfect equilibrium points of an extensive game with perfect recall. It is important to point out that it is not sufficient to analyse the ordinary normal form. This will be shown in section 13 with the help of a counterexample.

Substitute sequences:

Let $G = (\Pi_1, \ldots, \Pi_n; H)$ be a game in normal form. A mixed strategy q_i of player i is called *completely mixed* if for every $\pi_i \in \Pi_i$ the probability $q_i(\pi_i)$ assigned to π_i by q_i is positive. A mixed strategy combination $q = (q_1, \ldots, q_n)$ is called completely mixed if q_i is completely mixed for $i = 1, \ldots, n$. Let $\bar{q} = (\bar{q}_1, \ldots, \bar{q}_n)$ be a mixed strategy combination for G. An infinite sequence of mixed strategy combinations $q^1, q^2, \ldots$ is called a *substitute sequence* for $\bar{q}$ if q^k converges to $\bar{q}$ for $k \to \infty$ and every q^k is completely mixed. A strategy q_i or a strategy combination q is called a *best reply* to the substitute sequence $q^1, q^2, \ldots$ if q_i or q, resp. is a best reply to every q^k in the sequence.

Substitute perfect equilibrium points:

A mixed strategy combination $q^* = (q_1^*, \ldots, q_n^*)$ for a normal form G is called a *substitute perfect equilibrium point* of G if q^* is a best reply to at least one substitute sequence for q^*.

Lemma 10:

A substitute perfect equilibrium point of a normal form G is an equilibrium point of G.

Proof:

Let q^* be a best reply to the substitute sequence $q^1, q^2, \ldots$ for q^*. For $k = 1, 2, \ldots$ and for $i = 1, \ldots, n$ we have

$$H_i(q^k/q_i^*) = \max_{q_i \in Q_i} H_i(q^k/q_i). \qquad (57)$$

In view of the continuity of H_i and the continuity properties of the maximum operator it is clear that (57) remains valid if on both sides we take limits for $k \to \infty$. This shows that q^* is an equilibrium point.

Associated perturbed normal forms:

Let $G = (\Pi_1, \ldots, \Pi_n; H)$ be a normal form, let $q = (q_1, \ldots, q_n)$ be a completely mixed strategy combination for G and let ε be a positive number such that for

$i = 1, \ldots, n$ we have $q_i(\pi_i) > \varepsilon$ for every $\pi_i \in \Pi_i$. For every triple (G, q, ε) of this kind we define an associated perturbed normal form $\hat{G} = (G, \eta)$, where the minimum probabilities of the pure strategies for G are as follows:

$$\eta_{\pi_i} = \begin{cases} q_i(\pi_i) & \text{if } \pi_i \text{ is not a best reply to } q \text{ in } G \\ \varepsilon & \text{if } \pi_i \text{ is a best reply to } q \text{ in } G \end{cases} \tag{58}$$

for $i = 1, \ldots, n$ and for every $\pi_i \in \Pi_i$. Obviously η satisfies the condition that the minimum probabilities for all pure strategies of a player sum up to less than 1.

Lemma 11:

Let $\hat{G} = (G, \eta)$ be the associated perturbed normal form for the triple (G, q, ε). The strategy combination q is an equilibrium point of $\hat{G}$.

Proof:

A mixed strategy is a best reply to q in $\hat{G}$ if the pure strategies which are not best replies to q in G are used with their minimum probabilities. In view of (58) this is the case for every component q_i of q.

Lemma 12:

A substitute perfect equilibrium point of a normal form G is a perfect equilibrium point of G.

Proof:

Let $q^* = (q_1^*, \ldots, q_n^*)$ be a substitute perfect equilibrium point for G and let $q^1, q^2, \ldots$ be a substitute sequence for q^* such that q^* is a best reply to $q^1, q^2, \ldots$. Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive numbers with $\varepsilon_k \to 0$ for $k \to \infty$, such that for $k = 1, 2, \ldots$ and for $i = 1, \ldots, n$ we always have $q_i^k(\pi_i) > \varepsilon_k$ for every $\pi_i \in \Pi_i$. Since every q^k is completely mixed we can find a sequence $\varepsilon_1, \varepsilon_2, \ldots$ of this kind. Let $\hat{G}^k = (G, \eta^k)$ be the perturbed normal form associated with the triple (G, q^k, ε_k).

In the following it will be shown that $\hat{G}^1, \hat{G}^2, \ldots$ is a test sequence for G. Obviously for $k \to \infty$ those minimum probabilities which are equal to ε_k converge to 0. Consider a pure strategy $\pi_i \in \Pi_i$ which is not a best reply to q^*. For this pure strategy we must have $q_i^*(\pi_i) = 0$. Therefore for $k \to \infty$ the minimum probabilities of pure strategies which are not best replies to q^* converge to 0, too. Consequently, $\hat{G}^1, \hat{G}^2, \ldots$ is a test sequence of G.

The sequence $q^1, q^2, \ldots$ is a sequence of equilibrium points q^k for the perturbed game $\hat{G}^k$ in a test sequence $\hat{G}^1, \hat{G}^2, \ldots$ for G. This follows by lemma 11. Moreover the sequence $q^1, q^2, \ldots$ converges to q^*. Therefore q^k is a limit equilibrium point of the test sequence $\hat{G}^1, \hat{G}^2, \ldots$. Consequently q^k is a perfect equilibrium point of G.

Theorem 7:

A mixed strategy combination $q^* = (q_1^*, \ldots, q_n^*)$ is a perfect equilibrium point of G, if and only if q^* is a substitute perfect equilibrium point of G.

Proof:

In view of lemma 12 it remains to be shown that a perfect equilibrium point q^* of G is substitute perfect. Let $\hat{G}^1, \hat{G}^2, \ldots$ be a test sequence for G, such that q^* is a limit equilibrium point of $\hat{G}^1, \hat{G}^2, \ldots$. Let $q^1, q^2, \ldots$ be a sequence of equilibrium points q^k for $\hat{G}^k$ which converges to q^*. The definition of a perfect equilibrium point requires that such sequences $\hat{G}^1, \hat{G}^2, \ldots$ and $q^1, q^2, \ldots$ exist.

Let T_i^k be the set of all those pure strategies of player i which appear with more than minimum probability in q^k, i.e. π_i is in T_i^k, if and only if we have $q_i^k(\pi_i) > \eta_{\pi_i}^k$ for player i's component q_i^k in q^k. Obviously a pure strategy $\pi_i \in T_i^k$ is a best reply q^k in G but T_i^k may not contain every pure best reply to q^k in G. Since the q^k converge to q^* and the $\eta_{\pi_i}^k$ converge to 0, there must be a number m, such that for $k > m$ every pure strategy π_i with $q_i^*(\pi_i) > 0$ is in T_i^k for $i = 1, \ldots, n$. Without loss of generality we can assume $m = 0$ since otherwise we can use subsequences of the original sequences $\hat{G}^1, \hat{G}^2, \ldots$ and $q^1, q^2, \ldots$ for the purpose of this proof.

Since every π_i with $q_i^*(\pi_i) > 0$ is in T_i^k and every $\pi_i \in T_i^k$ is a best reply to q^k in G, the mixed strategy q_i^* is a best reply to g^k for $k = 1, 2, \ldots$. The q^k are completely mixed and $q^1, q^2, \ldots$ converges to q^*. The sequence $q^1, q^2, \ldots$ is a substitute sequence for q^* and q^* is a best reply to this sequence. q^* is a substitute perfect equilibrium point.

13. Two Counterexamples

One might be tempted to think that a perfect equilibrium point of the normal form G of an extensive game Γ with perfect recall always corresponds to a perfect equilibrium point of Γ. If this were the case one would not need the agent normal form. In the following we shall present two counterexamples. The first one is quite simple but less satisfactory than the second one.

The first counterexample:

The extensive game of figure 2 has exactly one perfect equilibrium point, namely the pure strategy combination (Rr, L). Here Rr refers to that pure strategy of player 1 where he chooses R at the origin and r at his other information set. The fact that this is the only perfect equilibrium point follows immediately by the subgame perfectness of perfect equilibrium points. (See the corollary of theorem 2 in section 8).

In the normal form (Rr, L) is a perfect equilibrium point, too but not the only one. Since the strategies Rl and Rr are equivalent (Rl, L) is just as perfect in the normal form as (Rr, L). In a perturbed game of the extensive form the strategies Rl and Rr are not equivalent but this information is lost in the normal form and cannot be regained by the construction of perturbed normal forms.

The first counterexample is not quite satisfactory since one may be content with the fact that among the two equivalent perfect equilibrium points of the

normal form, there is one which is perfect in the extensive form. One may take the point of view that it is not important to distinguish between these two equilibrium points.
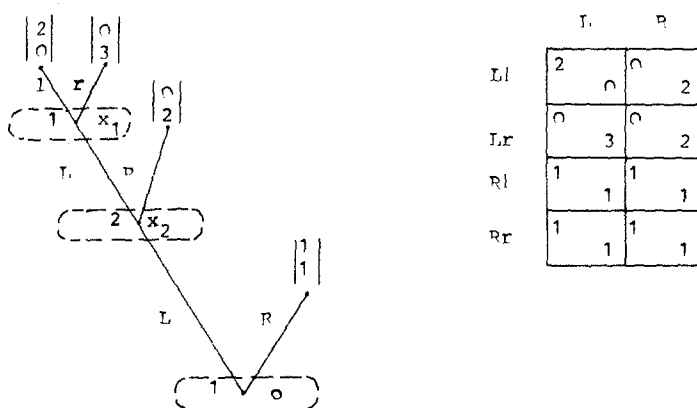

Fig. 2. Extensive form and normal form for the first counterexample. The conventions of the graphical representation of the extensive form are explained at figure 1

The second counterexample:

Consider the equilibrium points (Rl, L_2, R_3) and (Rr, L_2, R_3) of the game of figure 3. As we shall see both of these equilibrium points are perfect in the normal form but they fail to be perfect in the extensive form.

Perfectness in the normal form:

It is sufficient to show that (Rl, L_2, R_3) is a perfect equilibrium point of the normal form, if this is the case the same must be true for (Rr, L_2, R_3) since in the normal form Rr is a duplicate of Rl.

In order to show the perfectness of (Rl, L_2, R_3) we construct the following substitute sequence $q^1, q^2, \ldots$: In q^k every pure strategy which does not occur in (Rl, L_2, R_3) is used with a small probability ε_k. The pure strategies Rl, L_2 and R_3 are used with probabilities $1 - 3\varepsilon_k$, $1 - \varepsilon_k$ and $1 - \varepsilon_k$, resp. The ε_k are selected in such a way that $\varepsilon_1, \varepsilon_2, \ldots$ is a decreasing sequence which converges to 0. Of course ε_1 must be selected sufficiently small, say $\varepsilon_1 < 1/100$.

It can be verified easily that (Rl, L_2, R_3) is a best reply to this substitute sequence. We omit the computational details. It follows by theorem 7 that (Rl, L_2, R_3) and (Rr, L_2, R_3) are perfect equilibrium points for the normal form of the game of figure 3.

Imperfectness of (Rl, L_2, R_3) in the extensive form:

Let $b^1, b^2, \ldots$ be a sequence of behavior strategy combinations with completely mixed local strategies for the game in figure 3 such that b^k converges to (Rl, L_2, R_3) for $k \to \infty$. We may call $b^1, b^2, \ldots$ a *substitute sequence* for (Rl, L_2, R_3). According

to theorems 4 and 7 the equilibrium point (Rl, L_2, R_3) cannot be perfect unless there is at least one such sequence with the property that the choices in (Rl, L_2, R_3) are local best replies to every b^k. In order to see this one just has to translate the terminology of the agent normal form into that of the extensive form.

For every ε there is a number $m(\varepsilon)$ such that for $k > m(\varepsilon)$ the probabilities prescribed by b^k to the choices selected by (Rl, L_2, R_3) are greater than $1 - \varepsilon$. It can be seen immediately that for sufficiently small ε the local best reply for player 1 is r, not l. This shows that the sequence $b^1, b^2, \ldots$ cannot be such that (Rl, L_2, R_3) is a best reply to every b^k. Consequently (Rl, L_2, R_3) fails to be a perfect equilibrium point of the extensive game of figure 3.

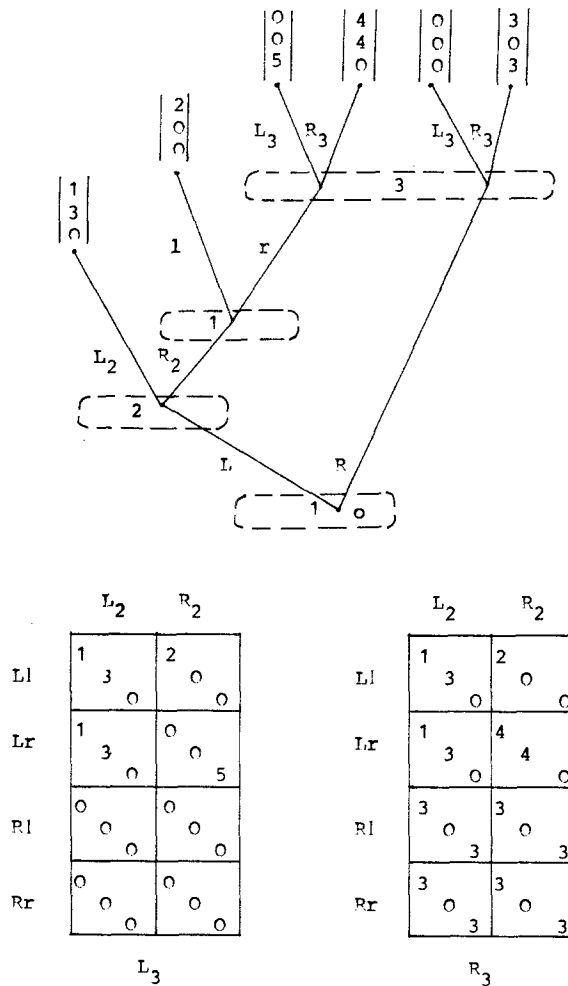

Fig. 3. Extensive form and normal form for the second counterexample. The normal form is described by two trimatrices, one for player 3's choice L_3 and one for his choice R_3

Imperfectness of (Rr, L_2, R_3) in the extensive form:

In the same way as before let $b^1, b^2, \ldots$ be a sequence of behavior strategy combinations with completely mixed local strategies which converges to (Rr, L_2, R_3). Here, too, for sufficiently big k the choices in (Rr, L_2, R_3) have probabilities greater than $1 - \varepsilon$ in b^k. It can be seen immediately that for sufficiently small ε player 2's best reply to b^k is R_2. Therefore the sequence $b^1, b^2, \ldots$ cannot be such that (Rr, L_2, R_3) is a best reply to every b^k. Consequently (Rr, L_2, R_3) fails to be a perfect equilibrium point of the game of figure 3.

Interpretation:

In the following we shall try to give an intuitive explanation for the phenomenon that an equilibrium point which is perfect in the normal form may not be perfect in the extensive form.

In order to compare the normal form definition with the extensive form definition, we shall look at a perturbed game $\hat{\Gamma}$ of an extensive game Γ with perfect recall and at a perturbed normal form $\hat{G}$ of the normal form G of Γ. Let the behavior strategy combination $b^k = (b_1^k, \ldots, b_n^k)$ be an equilibrium point for $\hat{\Gamma}$ and let the mixed strategy combination $q^* = (q_1^*, \ldots, q_n^*)$ be an equilibrium point for $\hat{G}$.

A choice c in Γ is called *essential* for b^* if the relevant local strategy selects c with more than the minimum probability for c required by $\hat{\Gamma}$. A choice which is essential for b^k must be a local best reply to b^k in Γ.

A pure strategy π_i is called *essential* for q^k if $q_i^*(\pi_i)$ is greater than the minimum probability for π_i required by $\hat{G}$. A pure strategy which is essential for q^* must be a best reply to q^* in Γ.

Both b^* and q^* reach all parts of the extensive form in the sense that the realization probabilities of all vertices are positive. Nevertheless there is a crucial difference between b^* and q^*. This difference concerns the conditional choice probabilities $\mu_i(c, u, q_i^*)$ which have been defined with the help of lemma 1 and lemma 2 in the proof of KUHN's theorem. In the case of q^* these conditional choice probabilities are defined for every personal information set.

It may happen that player i's pure strategies which are essential for q^* are such that a given information set u is not reached by q^*/π_i for every one of these essential strategies π_i; the realization probabilities $\rho(x, q^*/\pi_i)$ are 0 for every $x \in u$. An information set u of this kind will be called *inessentially reached* by q^*.

If an information set u of player i is inessentially reached by q^*, then the conditional choice probabilities $\mu(c, u, q^*)$ will be exclusively determined by those pure strategies of player i which are inessential for q^*. Therefore the $\mu_i(c, u, q^*)$ may be very unreasonable as a local strategy at u.

The crucial difference between b^* and q^* is as follows: Whereas every local strategy in b^* is reasonable in the sense that the essential choices are local best replies, q^k may lead to unreasonable conditional choice probabilities at those information sets which are inessentially reached by q^*.

As an example let Γ be the game in figure 3 and let q^* be such that only the pure strategies in the equilibrium point (Rr, L_2, R_3) are essential for q^*. The information set of player 1, where he chooses between l and r is inessentially reached. Therefore the conditional choice probabilities for l and r are not determined by Rr but exclusively by the minimum probabilities for Ll and Lr which may be such that l is selected with a high conditional choice probability.

In an extensive game, where every player has at most one information set, it cannot happen that the information set of a player i is not reached by q^*/π_i for one of his pure strategies π_i. His strategy does not influence the realization probabilities of the vertices in his information set. The agent normal form corresponds to an extensive form where every player has at most one information set. Therefore no difficulties arise in the agent normal form.

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